$$y_n = \frac{x - d_{n-1}(\tau)}{d_n(\tau) - d_{n-1}(\tau)}, \quad n = \overline{1, N}; \quad t = \tau.$$

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RESOLVING POWER OF THE ITERATION METHOD OF SOLVING INVERSE HEAT-CONDUCTION BOUNDARY-VALUE PROBLEMS

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UDC 536.24

A quantitative estimate is obtained for the range of frequencies entering in the boundary condition that is restorable by using a solution of the inverse problem by an iteration method.

Methods of solving inverse heat-conduction problems (IHCP) should possess smoothing properties that would not cause fluctuations of the solution. Such smoothing is assured, for instance, because of the natural step regularization [1], the introduction of extremal formulations of the IHCP and so-called stabilizing functionals [1, 2]. Regularized algorithms are used to seek the solution in a set of functions possessing a definite degree of smoothness, and suppress high frequencies in the parameters being recovered. However, if the fluctions in the desired characteristics are physical in nature, then the viscosity properties of the regularized algorithms do not permit detection of the fine structural features of the solution.

Therefore, when solving the IHCP a situation must be met when the "noninertial" unregularized algorithms pass high frequencies, but because of incorrectness there is no possibility of clarifying the physical component among them and regularization does not afford such a possibility because it filters high harmonics independently of their origin. There, there-, fore, arises the problem of determining the range of frequencies in the restorable parameters as a preliminary step in the selection of methods of raising the accuracy of solving the IHCP, especially in the case of complex behavior of the desired functions. Increasing the measurement accuracy, taking account of a priori information [1], rational placement of the temperature sensors in the object under investigation with application of the Fisher information matrix [3] can be such methods.

Two reasons for suppression of the high frequencies are represented essential for the solution of inverse problems: the smoothing action of the heat-conduction operator and the discretization of the continuously formulated problem.

To obtain quantitative estimates of the passband, we consider the model of a semiinfinite body with thermal diffusivity coefficient a. As is noted in [1], the smoothing action of the heat conduction operator can be estimated by giving the change in body surface temperature according to a sinusoidal law $T_W = T_0 \sin(\omega\tau)$. Then after a certain time the temperature at a depth h will also be described by a sinusoid [4]

$$T(h, \tau) = T_h \sin(\omega\tau - \varphi), \tag{1}$$

where φ is a certain phase difference.

The amplitude of the oscillations T_h is defined as follows

$$T_h = T_0 \exp\left(-\sqrt{\frac{\omega}{2a}}\, h\right). \tag{2}$$

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Introducing the dimensionless frequency $\tilde{\omega} = \omega h^2/a$, we have

$$k = \exp\left(-\sqrt{\frac{\tilde{\omega}}{2}}\right) \tag{3}$$

(see Fig. 1). Considering the quantity ε the relative error in measuring the temperature, we obtain an expression for the value of the critical frequency $\tilde{\omega}_{cr1}$:

$$\tilde{\omega}_{cr1} = 2\ln^2 \varepsilon. \tag{4}$$

It is natural to consider that all frequencies $\tilde{\omega} \leqslant \tilde{\omega}_{cr1}$ penetrate to a depth h and are detected by a sensor, while the higher ones are delayed. Therefore, frequencies from the interval $|0.2\ \ln^2\varepsilon|$ will be present in the parameter restored by means of solving the IHCP, while the higher harmonics turn out to be unrestorable.

Moreover, since straight lines are used in each iteration in the gradient method of solving the IHCP, higher harmonics from this interval turn out to be attenuated in some measure or other.

Another constraint on the upper boundary of the passband is, as already noted, due to discretization of the time interval for the numerical realization of the IHCP (natural step regularization). If it is considered that the least period of fluctuations corresponds to four time steps Δτ (Fig. 2), then from the condition

$$4\omega\Delta\tau \leqslant 2\pi \tag{5}$$

the constraint

$$\tilde{\omega}_{cr2} = \frac{\pi h^2}{2a\Delta\tau} \tag{6}$$

can be obtained.

Results of restoring the heat flux density deliverable to the boundary of a semiinfinite body from the solution of an IHCP in an external formulation are presented below.

The entrance temperature f(τ) was given in the form of a Duhamel integral

$$f(\tau) = \int_0^\tau \frac{a}{\lambda} q_{M}(\xi)\, G(h,\ 0;\ \tau - \xi)\, d\xi, \tag{7}$$

where $q_M(\tau)$ is the model (real) heat flux density on the boundary, and G(x, y; τ) is the Green's function of the boundary-value problem for a semiinfinite body [5]:

$$G(x,\ y;\ \tau) = \frac{1}{2\sqrt{\pi a\tau}}\left\{\exp\left[-\frac{(x-y)^2}{4a\tau}\right] + \exp\left[-\frac{(x+y)^2}{4a\tau}\right]\right\}. \tag{8}$$

The solution of the IHCP reduces to searching for the function q(τ) that minimizes the residual

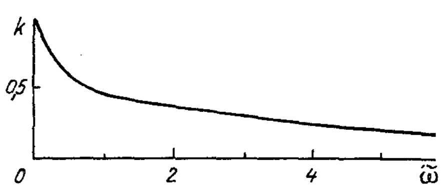

Fig. 1

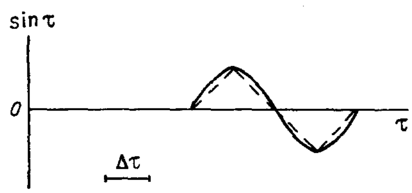

Fig. 2

Fig. 1. Dependence of the amplitude attenuation factor on the dimensionless fluctuation frequency.

Fig. 2. Restoration of one sinusoid wave by means of four discretization steps.

TABLE 1. Dependence of the Quantities k and η on the Dimensionless Frequency $\tilde{\omega}$ for the 100th Iteration

$\tilde{\omega}$	0	1	2	3	4	5	6	7	8	9
k	1	0,49	0,37	0,29	0,24	0,21	0,18	0,15	0,13	0,12
η	1	1	0,99	0,98	0,94	0,83	0,69	0,55	0,42	0,33

$$J(q)=\int_0^{\tau_m}[T(q,\ h,\ \tau)-f(\tau)]^2\,d\tau, \tag{9}$$

where

$$T(q,\ h,\ \tau)=\int_0^{\tau}\frac{a}{\lambda}\,q(\xi)\,G(h,\ 0;\ \tau-\xi)\,d\xi \tag{10}$$

is the solution of the direct problem of heat conduction to which the heat-flux density q(τ).

According to the method of conjugate gradients, the iteration process for approaching the minimum point of the functional (9) is realized as follows [1]

$$\overset{(n+1)}{q}(\tau)=\overset{(n)}{q}(\tau)-\beta_n S_n(\tau),\ n=0,\ 1,\ 2,\ \ldots, \tag{11}$$

$$S_n(\tau)=\overset{(n)}{J'_q}(\tau)-\gamma_n S_{n-1}(\tau), \tag{12}$$

$$\gamma_n=\frac{\int_0^{\tau_m}[\overset{(n)}{J'_q}(\tau)]^2\,d\tau}{\int_0^{\tau_m}[\overset{(n-1)}{J'_q}(\tau)]^2\,d\tau},\quad \gamma_0=0. \tag{13}$$

The optimal depth of the step at the n-th iteration was found from the condition $J(\overset{(n)}{q}-\beta S_n)=\min_\beta$:

$$\beta_n=\frac{\int_0^{\tau_m}[T(\overset{(n)}{q},\ h,\ \tau)-f(\tau)]\,\Delta T(\tau)\,d\tau}{\int_0^{\tau_m}\Delta T^2(\tau)\,d\tau}, \tag{14}$$

where ΔT is the solution of the direct problem of heat conduction in increments:

$$\Delta T(\tau)=\int_0^{\tau}\frac{a}{\lambda}\,S_n(\xi)\,G(h,\ 0;\ \tau-\xi)\,d\xi. \tag{15}$$

The gradient of the functional at the n-th iteration was determined by means of the formula

$$\overset{(n)}{J'_q}(\tau)=2\int_{\tau}^{\tau_m}[T(\overset{(n)}{q},\ h,\ \xi)-f(\xi)]\,G(h,\ 0;\ \xi-\tau)\,d\xi. \tag{16}$$

A program was compiled in the language PL/1 for the numerical realization of the algorithm. The following values were taken for the parameters in the computations: $\lambda = 10$ W/m·°K, $a = 10^{-6}$ m²/sec, $h = 10^{-3}$ m.

Illustrated in Fig. 3 is the resolution of the iteration method. The magnitude of the time step Δτ was taken equal to 0.1 sec, which, according to condition (6), eliminates the influence of the natural step regularization on the smoothing action of the algorithm if harmonics with frequencies $\tilde{\omega} \leqslant 15.7$ are present in the solution. The model heat flux density was given in the form $q_M(\tau) = q_0[1 + \sin(\omega\tau + \alpha)]$, where α, q_0 are constants. As is seen from Fig. 3, in this case the restoration of the heat flux density depends on the time

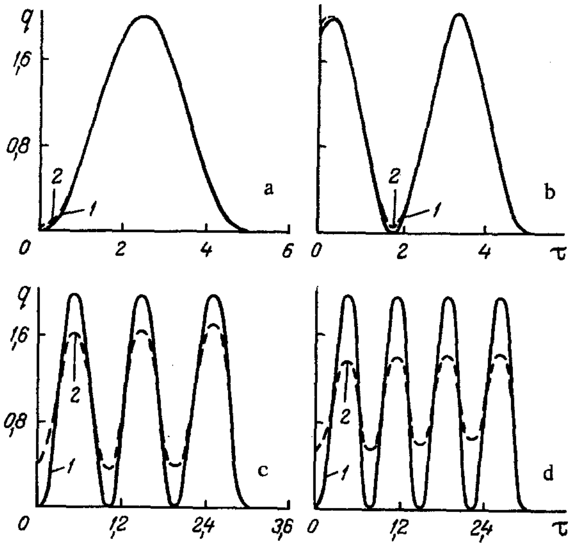

Fig. 3. Results of solving the IHCP on restoring the heat flux density with $\tilde{\omega}$ = 1.26 (a), 2.00 (b), 6.28 (c), 8.38 (d) (undisturbed initial data: 1) actual solution; 2) restored solution (100th iteration). q, 10^6 W/m², τ, sec.

$q(\tau) = q_0[1 + \eta(\omega) \sin(\omega\tau + \alpha)]$, where as the frequency ω grows attenuation of the appropriate harmonic occurs. Just as the parameter k(ω), governing the smoothing properties of the heat conduction operator, the function η(ω), that characterizes the "viscosity" of the interaction algorithm for solving the IHCP, decreases as ω grows but rather differently than k. Certain values of these functions are presented in the table.

Therefore, if the actual heat flux density $q_M(\tau)$ is characterized by the spectral density of the distribution Q(ω) [6]:

$$q_{\text{M}}(\tau) = \frac{2}{\sqrt{\pi}} \int\limits_0^\infty Q(\omega) \cos(\omega\tau)\, d\omega, \tag{17}$$

then its restoration by using the iteration method of minimizing the functional will result in the appearance of a filtering factor η(ω) in the spectral density Q:

$$q(\tau) = \frac{2}{\sqrt{\pi}} \int\limits_0^{\omega_{\text{кр}}} \eta(\omega) Q(\omega) \cos(\omega\tau)\, d\omega. \tag{18}$$

Presented in Fig. 4 are results of restoring a model heat flux density $q_M(\tau) = q_0[1.33 + \cos(1.26\tau) + \cos 0.4\,(7.67\tau)]$ with the perturbed input data $f_\delta = f + \Delta f$, where Δf is a random variable distributed in the segment $[-0.08f_{max}, 0.08f_{max}]$ according to a uniform law. It is seen that the halt of the iteration process according to the condition of matching the residual to the error in the input information results in suppression of the higher harmonics in the solution which can be restored because of diminution in the effect of noisiness, i.e., raising the accuracy of measuring the temperature.

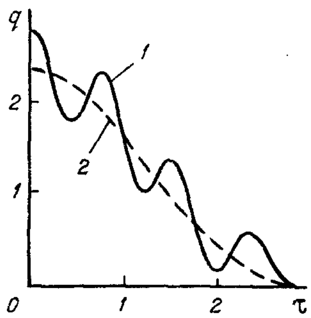

Fig. 4. Suppression of the high-frequency harmonics in solving the IHCP (input data perturbed according to a uniform law): 1) actual solution; 2) 4th iteration (half of the iteration process according to the residual criterion).

NOTATION

T, design temperature; f, input temperature; τ, time; $\Delta\tau$, time step; ω, circular frequency; $\tilde{\omega}$, dimensionless circular frequency; k, fluctuation amplitude attenuation coefficient: q, heat-flux density on the body boundary; a, thermal diffusivity factor; λ, heat-conduction coefficient; h, distance from the body surface to the point of temperature measurement; G, Green's function; J, functional; J_q', gradient of the functional; S_n, direction of descent in the n-th iteration, and η, viscosity index of the iteration algorithm.

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REGULARIZATION OF THE SOLUTION OF THE INVERSE HEAT-CONDUCTION PROBLEM IN A VARIATIONAL FORMULATION

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A version of the solution allowing numerical minimization of the target functional to be eliminated is considered.

The effectiveness of a combination of analytical and numerical methods of solution for the analysis of inverse heat-conduction problems (IHP) is a result of many factors. One of the most significant is analytical analysis, which largely determines the algorithm for solution of the IHP as a whole. In this respect, the example of using gradient methods to solve IHP is illustrative [1]. Finding the analytical expression for the gradient of the functional which eliminates the operation of numerical differentiation markedly expands the region of application of the algorithm developed. At present, there is an extensive bibliography on IHP solution; see [2], for example. However, despite the wealth of literature sources, the development of effective and simple computational algorithms even for one-dimensional IHP remains an urgent problem. This is associated with the multiplicity of IHP formulations sometimes requiring separate approaches, the increase in the demands on the accuracy of the results obtained, the appearance of new computational techniques permitting modeling at a qualitatively new level, and so on.

Now consider a version of the regularization of IHP solution in a variational formulation, which combined numerical and analytical methods of analysis and allows a sufficiently simple algorithm for boundary-condition identification to be obtained.

The basis of the approach is to establish the relation between the conditions defining the IHP and the desired boundary conditions [3]. For example, insolving IHP for a plane wall, this relates the known temperature and its gradient at one boundary to the desired temperature or its gradient at the other boundary.

In general form, the variational formulation of IHP was given in [1].

Defining the target functional analogously gives

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